3.16 Mathematics: Partial Fractions, Linear Law

Part one: Partial fractions

Partial fraction questions are perhaps the most direct kind of questions in your mathematics papers. It requires the least amount of brainwork when doing, and all that is for you to know is how to apply the cover-up method and what fractions to split into.

The basic idea of partial fractions is to split one fraction into sum or differences of simpler fractions. This idea is extremely useful in many areas, such as in differentiation/integration.

Step 1: Simplify expression by removing whole number parts and factorizing the denominator.

Some partial fraction questions trolled a lot of students when the numerator has the same if not higher degree as the denominator. Here the word "degree" refers to the *highest power* of the expression. For instance, in this expression:

$$\frac{2x^3+x^2-3x+1}{x^3-x}$$

One will be unable to split it into simpler fractions if you do not pull out the 2 from the numerator. That is, you have to do this:

$$\frac{2x^3+x^2-3x+1}{x^3-x}=2+\frac{x^2-x+1}{x^3-x}$$

Next, one has to *factorise the denominator* to be able to identify which fractions to split into. That is:

$$\frac{2x^3+x^2-3x+1}{x^3-x}=2+\frac{x^2-x+1}{x^3-x}=2+\frac{x^2-x+1}{x(x+1)(x-1)}$$

Step 2: Identify the form of the fractions it can split into.

Here we present a few common forms:

Examples:

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\frac{1}{(x-2)(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

Step 3: Apply cover-up method and coefficient comparison to arrive at *A, B, C*.

Cover-up method has one basic intuition: when you cross multiply over one fractor from the denominator of the LHS, by setting *x* to be the zero point of one of the factors, only the fraction with that factor as denominator will not be zero. This means:

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

If we multiply both sides by $(x+1)$, we have

$$\frac{1}{x(x-1)} = (x+1)\left[\frac{A}{x} + \frac{B}{x-1}\right] + C$$

Now notice that if we set $x = -1$ such that $(x+1) = 0$, we have

$$\frac{1}{(-1)(-2)} = 0 + C$$

This means we can simply look at $\frac{1}{x(x-1)}$ and set x to be -1 to find the value of C.

Similarly, we set $x = 0$ in $\frac{1}{(x+1)(x-1)}$ to get the value of A, etc.

For coefficient analysis, we are usually interested in the coefficient of the highest or lowest power (that is, the constant term).

$$\frac{1}{(x-2)(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

Given this, by cover-up method we have B = $-\frac{1}{3}$ and $C = \frac{1}{9}$ so all that is left is to find A. Notice that the coefficient of x^3 on the numerator of the LHS is 0.

Now B contributes 0 and C contributes $\frac{1}{9}$, thus A has to contribute $-\frac{1}{9}$ in order to form 0. Thus A is $-\frac{1}{9}$

IMPORTANT NOTE:

You have to understand the intuition behind this; not everytime you can simply take the value of B and C as what it contributes. For instance,

$$\frac{1}{(2x-1)(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x-1}$$

For this example, by cover-up B is $-\frac{1}{3}$ and C is $\frac{4}{9}$, so how to determine the value of A by coefficient analysis? Does B contribute $-\frac{1}{3}$ and C contribute $\frac{4}{9}$? NO! In this case, B actually contributes $-\frac{2}{3}$. The idea behind coefficient analysis is that we consider a portion of the hardcore combination of the fractions. That is, the RHS is actually

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x-1} = \frac{A(x+1)^2(2x-1) + B(x+1)(2x-1) + C(x+1)(x+1)^2}{(x+1)^2(x+1)(2x-1)}$$

As can be seen, in the expansion of the each of the three summands on the top, the coefficient of the x^3 term for the first is 2A, for the second is 2B and for the third is C. Thus, the second term actually contributes 2B, which is $-\frac{2}{3}$ here.

Part two: Linear Law

Why do we need linear law? Consider a physics experiment, where you control a variable *x* and you measure a variable *y*. You guess that the formula that gives the relationship between *x* and *y* to be a certain equation, but you have several constants. The *easiest* way to arrive at these constants is to change the equation into a *linear one*.

You have several tools in your arsenal to help you with this task.

1. Taking logs. In order to take log on both sides, most of the time you need both sides to be made up of *only products*. This is because it is easy

to split $\log xy = \log x + \log y$, but it is almost impossible to do anything with $\log(x + y)$. It is easy to spot that you have to use logs when your variables are in the *exponents*.

Example: $ax^2 = b^2 \frac{y}{x}$ taking log on both sides give $\log a + 2\log x = 2\log b + \log y - \log x$ which becomes $\log y = 3\log x + (\log a - 2\log b)$ It is overkill for this question, but it's just for demonstration.

2. Factorisation. Typically, when the equation is not made up of products, i.e. both sides are sum of several terms, you will need to factorise it. There will be several factors made up of your variable and *known constants*. Example: (x+2) or (2y-3). But there will be ONE factor with your unknown constants, i.e. (ax+b) or (ab+xy). This is the one you leave on the RHS, while you move EVERYTHING else over to the LHS.

   Example: $\frac{hx}{y^2} - k = x^2 - (k-1)x$
   Moving the *k* over and factorizing we have $\frac{hx}{y^2} = (x-k)(x-1)$. There are many ways to continue, for example dividing (x-1) over we have
   $$\frac{x}{y^2(x-1)} = \frac{1}{h}x - \frac{k}{h}$$

3. Completing the square; quadratic simplification. This means to turn something like $x^2 + 2kx + 3$ into $(x+k)^2 + (3 - k^2)$.

   Example: $y^2 = x^2 + x + 3a$
   Completing the square on the RHS gives
   $$y^2 = (x + \frac{1}{2})^2 + (3a - \frac{1}{4})$$
   This alone if you notice is already of linear form.

I would like to emphasise once again that math is a subject that has no substitution for practice and working out problems. What I am doing here is to give you some ideas that you can work with. The process of doing math problems is to keep trying different ideas until you eventually arrive at the answer with one method.

Exercises:

Split the following into partial fractions

1. $\frac{1}{x(x+2)}$

2. $\frac{1}{(x-1)(x+5)}$

3. $\frac{1}{x(x+2)(3x+1)}$

4. $\frac{3x^3-2x+1}{x^3-2x^2+x}$

5. $\frac{2x+7}{x^3+6x^2+5x+6}$

Convert the following into linear expression Y = AX + B, where X, Y is made up of variables *x, y* and known constants while A, B are made up of known and unknown constants. Warning: these questions are meant to destroy you.

Equation	Linear form	Y	X	A	B
$b^{xy} = xa^3$					
$y = kx^2 - x$					
$y^2 + xy$ $= ax^2 - xy + b$					
$a(x+y) - b$ $= x^2 + (b+1)x$					
$x - k = yx - yk + 5$					

For the first time from me…

Answers:

1. $\frac{1}{x(x+2)} = \frac{1}{2x} - \frac{1}{2(x+2)}$
   Direct application of cover-up method.
2. $\frac{1}{(x-1)(x+5)} = \frac{1}{6(x-1)} - \frac{1}{6(x+5)}$
   Direct application of cover-up method.
3. $\frac{1}{x(x+2)(3x+1)} = \frac{1}{2x} + \frac{1}{10(x+2)} - \frac{9}{5(3x+1)}$
   Direct application of cover-up method.
4. $\frac{3x^3-2x+2}{x^3-2x^2+x} = 3 + \frac{2}{x} + \frac{3}{(x-1)^2} + \frac{1}{x-1}$
   First take out the 3 and factorise the denominator. Then apply cover-up and coefficient analysis.
5. $\frac{2x+7}{x^3+6x^2+5x+6} = \frac{5}{2(x+1)} - \frac{3}{x+2} + \frac{1}{2(x+3)}$
   Factorise denominator, then apply cover-up.

Equation	Linear form	Y	X	A	B
$b^{xy} = xa^3$	$xy = \frac{1}{\log b}(\log x) + \frac{3}{\log b}(\log a)$	x^2	$\log x$	$\frac{1}{\log b}$	$\frac{3}{\log b}(\log a)$
$y = kx^2 - x$	$\frac{y}{x} = kx - 1$	$\frac{y}{x}$	x	k	-1
$y^2 + xy = ax^2 - xy + b$	$(x+y)^2 = (a+1)x^2 + b$	$(x+y)^2$	x^2	$(a+1)$	b
$a(x+y) - b = x^2 + (b+1)x$	$\frac{x+y}{x+1} = \frac{x}{a} + \frac{b}{a}$	$\frac{x+y}{x+1}$	x	$\frac{1}{a}$	$\frac{b}{a}$
$x - k = yx - yk + 5$	$\frac{5}{1-y} = x - k$	$\frac{5}{1-y}$	x	1	$-k$

1. Take log on both sides, then divide throughout by log b.
2. Factorise the RHS, dividing over the x to the LHS.
3. Add xy+x^2 to both sides of the equation, then factorise the LHS (it's a perfect square!)
4. Add b to both sides of the equation, then factorise the RHS, dividing throughout by (x+1)
5. Move all terms except the 5 from RHS to LHS, factorise then divide (1-y) over to the RHS.