Mathematics: Exponential, Logarithmic and Quadratic Functions

Exponential and Logarithmic functions

Introduction:

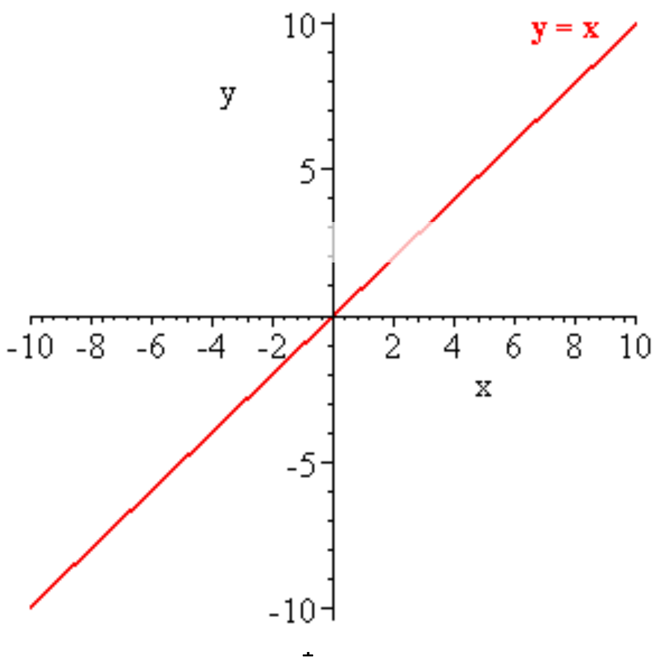
In mathematics, the concept of function is extremely important. So before we go into anything, we will first go through what a function is.

Taking one thing, turning it into another.

This is essentially the entire concept of a “function”. Think of a machine that takes in a number, and churns out another. This is what a function is. How the machine churns out this number depends on how the machine is “defined”.

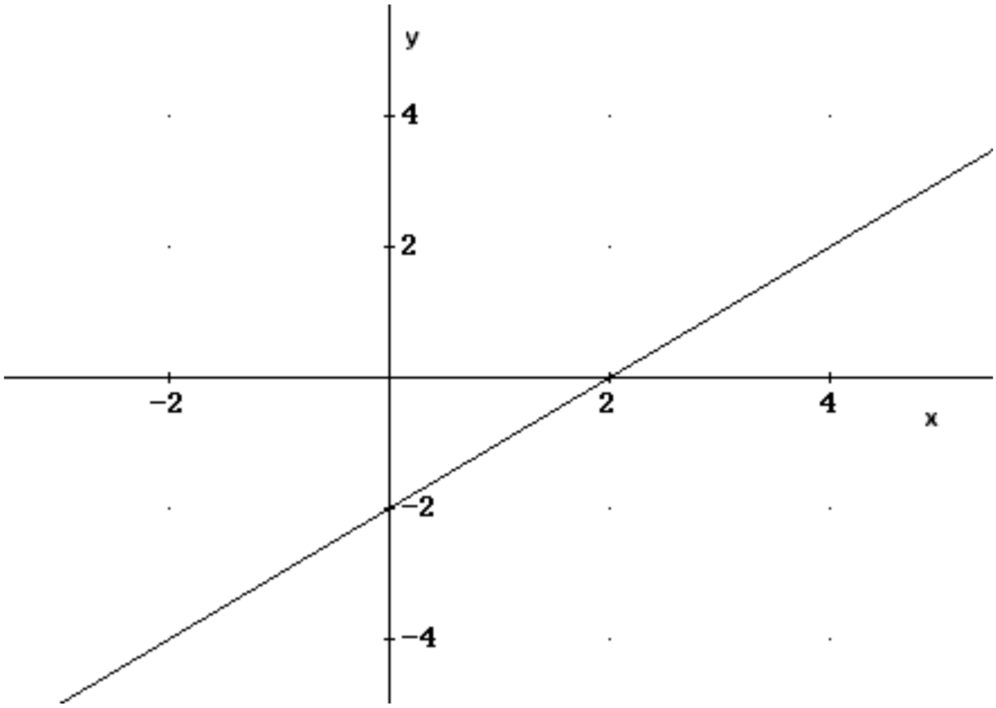
Graphing functions.

Graphing is a useful way to represent functions. One may of course write out the function in terms of an algebraic expression in *x*, but graphing provides a pictorial representation of functions. Let’s take the function f(x)=x. Plotting f(x) as y against x,



As can be seen, we have a line. If we were to *replace all x with (x-a)*, what we will be effectively doing is *shifting the whole graph a units to the RIGHT* ***without changes in the vertical direction.***

For this easy example, this is the graph of f(x) = x-2



Do remember that minus corresponds to a movement to the *RIGHT.* This should not be a problem, you can simply check by putting in *x=0*, and checking where this value is.

Appending a constant *b* to the end will shift the graph up/down. Think of it this way: y = f(x), now replace y with (y-b), this will achieve the same effects, shifting the graph up/down.

Formulas for exponential and logarithmic functions:

$$a^x \times a^y = a^{x+y}$$

$$a^x \div a^y = a^{x-y}$$

$$(a^x)^y = a^{xy}$$

$$\sqrt[y]{a^x} = a^{\frac{x}{y}}$$

$$log_a x + log_a y = log_a xy$$

$$log_a x - log_a y = log_a \frac{x}{y}$$

$$log_a x^b = b \, log_a x$$

$$log_{a^b} x = log_a x^{\frac{1}{b}}$$

$$\frac{1}{log_a x} = log_x a$$

There are only two types of questions in this topic: Graph and solving equation.

Graph Drawing

Important notes:

1. For exponential and logarithmic graphs, do remember to draw in your asymptotes. They should *never* touch any part of your graph. For an exponential graph, the equation of the asymptote should be in the form of *x = a*, where *a* is typically 0 but can be shifted up or down.
2. For Logarithmic graphs, the asymptote should be in the form of *y = b*, where b is typically 0, but can be shifted up or down.
3. Do mark out the x-intercept and y-intercept. For graph e^x, the y-intercept should be 1. For graph $\ln x$, the x-intercept should be 1. When in doubt, let *y = 0* to check for x-intercept and let *x = 0* to check for y-intercept.
4. Remember to write the equation of ALL your lines.
5. Recall how to shift graphs around, as shown in first section.

Exercise

Graph the following equations:

a) $y = 5e^{2x-1} + 2$
b) $y = -\ln(x - 5) + 3$
c) $y = e^{-x} - 2$

Solving Exponential and Logarithmic functions

First and foremost, if applicable, simplify ALL the logs. This means to apply

$$log_a x^b = b\, log_a x$$

To simplify all expressions such as $\log_3 \sqrt{2x^2 + 1}$ into $\frac{1}{2}\log_3(2x^2 + 1)$, AND apply

$$log_a x + log_a y = log_a xy$$

To split *factorisable quadratic expressions* such as

$$\log_2(x^2 - 5x + 6).$$

Simplification is left as an exercise to the reader.

Next, if applicable, change ALL logs to be of the SAME BASE, by applying

$$\frac{1}{log_a x} = log_x a$$

This is where you have to look at the equation and decide which expression you are to use as the base. Usually, we choose a constant.

From here, most questions require you to use substitution, as a single expression will be repeated throughout. Usually this results in a quadratic equation.

If the algebraic expressions are different from one another, another common method requires you to *combine terms* after changing to same base using $log_a x + log_a y = log_a xy$. How this works is that you have

log (something) = log (something else)

This implies

(something) = (something else).

REMEMBER: A constant be changed into log. E.g. $1 = \log_7 7$, etc.

For equations with exponential expressions, observe and choose the most convenient substitution, then it usually becomes a quadratic equation.

Exercise

Solve the following equations.

a) $e^{2x-2} = 2e^{x-1} + 3$
b) $\log_3 \sqrt{2x^2 + 1} = 1 + \log_9(2x - 3)$
c) $2^{x+4} - 2^{-x} = -15$

Do not forget to solve for your substitution at the end, and sub back in to check (in particular, the quadratic equations with negative solutions).

Part two: Quadratic Functions

All *a, b, c and x* below refer to the coefficients/variable of the quadratic equation $ax^2 + bx + c = 0$, discriminant refers to $(b^2 - 4ac)$.

Formulas and important results:

a. If *a > 0*, the graph of the function extends *upwards.*
b. If *a < 0,* the graph of the function extends *downwards.*
c. If discriminant < 0, the graph never touches the *x-axis.*
d. If discriminant > 0, the graph touches the *x-axis at two points.*
e. If discriminant = 0, the graph is tangent to the *x-axis*
f. Combining the above we have:
   i. If *a > 0,* discriminant < 0, the function is **always positive.**
   ii. If *a < 0,* discriminant < 0, the function is **always negative.**
g. If given linear function $y = px + q$, it will intersect with the quadratic equation at
   i. One point; i.e. tangent to the quadratic curve if there is **only one solution to the simultaneous equation**.
   ii. Two points if there are **two solutions to the simultaneous equation.**
   iii. 0 points; i.e. *does not meet the curve* if the simultaneous equation has no solutions.
h. To solve the above *simultaneous equation*, we can either
   i. Make y the subject in the linear equation; substitute this into the quadratic equation
   ii. Or make y the subject in both equations and equate the expressions for *x*

To check for number of solutions of a quadratic equation, we look at the discriminant.

Discriminant > 0 implies two real solutions.

Discriminant < 0 implies no real solutions.

Discriminant = 0 implies one real solution.

Do realize that quadratic inequalities come out often for this topic. Pls review.

Exercises

1. Find the greatest prime value of *p* for which $x^2 + 6x + 25 - 2px$ is always positive for all values of *x.*

2. (i) Find the range of values of *m* such that the line $y = mx + 2$ does not meet the curve $x^2 + y^2 - 3x + 1 = 0$

   (ii) Show that if $k \geq 5$, the line $3x + 2y = 5$ meets the curve $3x^2 + 2y^2 = k$ for all real values of *x*.

3. a) Find the range of values of *x* for which $2x(x+3) - x - 3 > 0$

   b) Find the range of values of *p* for which $\frac{2x^2 - 3px + 1 - \frac{3p}{4}}{-2} < 0$

4. Find the range of values of *k* for which the graph of $y = (k+1)(x^2 + 1) - kx$ lies entirely below the *x-axis.*