Mathematics EOY Notes

Topic 2: Surds

Worksheet 1

Summary

- Learn the basics of Surds
- Simplify expressions in Surds

Formulae

→ $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
→ $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
→ $\sqrt{a} \times \sqrt{a} = a$
→ $a\sqrt{c} \times b\sqrt{d} = ab\sqrt{cd}$
→ $a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$

Example

a) $\sqrt{16} \times \sqrt{12} \times \sqrt{75} = 4 \times 2\sqrt{3} \times 5\sqrt{3}$ (Change all the terms into same surd)
$= 40\sqrt{9}$
$= 120$

a) $\sqrt{28} - \sqrt{175} + \sqrt{112} = 2\sqrt{7} - 5\sqrt{7} + 4\sqrt{7}$ (Change all the terms into same surd)
$= \sqrt{7}$

Worksheet 2

Summary

- Rationalise Surd expressions
- Simplify Surd expressions
- Solve Surd equations

Formulae

→ $\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$

→ $\frac{a}{b+\sqrt{c}} = \frac{a}{b+\sqrt{c}} \times \frac{b-\sqrt{c}}{b-\sqrt{c}} = \frac{ab-a\sqrt{c}}{(b+\sqrt{c})(b-\sqrt{c})} = \frac{ab-a\sqrt{c}}{b^2-c}$

Examples

a) $\sqrt{176}-\sqrt{99}+\frac{242}{\sqrt{44}} = 4\sqrt{11}-3\sqrt{11}+\frac{242}{2\sqrt{11}}$ (Change all terms into same surd)

$$= \sqrt{11}+\frac{242\sqrt{11}}{22}$$
$$= \sqrt{11}+11\sqrt{11}$$
$$= 12\sqrt{11}$$

b) $\sqrt{x+3}+\sqrt{x+8} = 5$ (Square both sides)

$$x+3+2(\sqrt{x+3})(\sqrt{x+8})+x+8 = 25$$
$$2x+11+2\sqrt{x^2+11x+24} = 25$$
$$2\sqrt{x^2+11x+24} = 14-2x \text{ (Square both sides again)}$$
$$4x^2+44x+96 = 4x^2-56x+196$$
$$100x = 100$$
$$x = 1$$

Topic Summary

- When doing surd questions, always factorize and make surds into the simplest form
- By doing so, usually you will end up with common factors in surds
- Also, don't forget to always rationalize a fraction by removing the surd term from the denominator

<u>Topic 3: Remainder and Factor Theorems, Partial Fractions</u>

<u>Worksheet 1</u>

Summary

- Write polynomial in Dividend = Divisor × Quotient + Remainder
- Distinguish between identities and equations
- Apply long or synthetic division to polynomials

Formulae

→ Dividend = Divisor × Quotient + Remainder

In general, we can divide a polynomial $P(x)$ by a polynomial $D(x)$ which is of the same or lower degree.

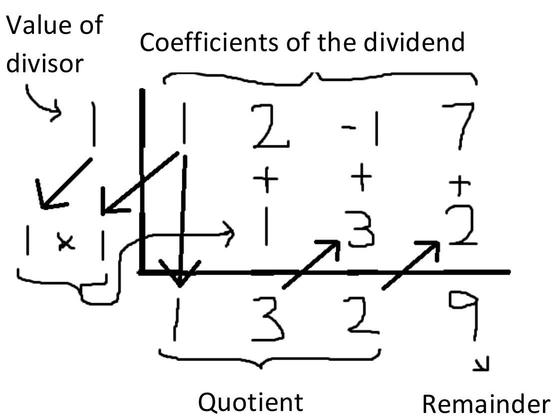
$$P(x) = D(x) \times Q(x) + R(x)$$

where $P(x)$ is the dividend, $D(x)$ is the divisor, $Q(x)$ is the quotient and $R(x)$ is the remainder.

→If $P(x) = Q(x)$ is true for all values of x, it is an identity.

→Synthetic Division (ONLY for Linear Divisors)

1) Write coefficients of the dividend and the value of the divisor. Remember to put 0 for missing terms.
2) Copy the coefficient of the first term
3) Multiply the coefficient of the first term to the divisor and write it below the coefficient of the second term
4) Add the coefficient of the second term to the product
5) Repeat the process



Examples

a) $(x^3 + 2x^2 - x + 7) \div (x - 1)$

$$\begin{array}{r|l} & x^2 + 3x + 2 \\ x-1 & x^3 + 2x^2 - x + 7 \\ & \underline{x^3 - x^2} \\ & \quad 3x^2 - x \\ & \quad \underline{3x^2 - 3x} \\ & \qquad 2x + 7 \\ & \qquad \underline{2x - 2} \\ & \qquad\quad 9 \end{array}$$

$(x^3+2x^2-x+7)=(x-1)(x^2+3+2)+9$

b) Find the values of A, B and C if $4x^2+3x-7 \equiv A(x-1)(x+3)+B(x-1)+C$

Method 1

Compare coefficients of x^2

$A=4$

Compare coefficients of x

$4(-1+3)+B=3$

$8+B=3$

$B=-5$

Compare coefficients of constant

$4(-1\times 3)+(-5\times -1)+C=-7$

$-12+5+C=-7$

$C=0$

Method 2

When $x=1$, $4+3-7=0+0+C$

$C=0$

When $x=-3$, $36-9-7=0+B(-4)+0$

$20=-4B$

$B=-5$

When $x=0$ $-7=A(-1)(-3)+(-5)(-1)+0$

$-7=-3A+5$

$3A=12$

$A=4$

$\therefore A=4, B=-5, C=0$

Worksheet 2

Summary

- Use the Remainder and Factor Theorem

Formulae

→If $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$

→If $f(x)$ is divided by $(ax-b)$, the remainder is $f(\frac{b}{a})$

→If $(ax-b)$ is a factor of $f(x)$, then $f(\frac{b}{a})=0$

→If $f(\frac{b}{a})=0$, then $(ax-b)$ is a factor of $f(x)$

Examples

a) The expressions x^3-7x+6 and $x^3-x^2-4x+24$ have the same remainder when divided by $x+p$. Find the possible values of p.

When ÷ by $x+p$,

Remainder : $f(-p)$

$f(x)=x^3-7x+6$

$-p^3-7p+6$--- (1)
$f(x)=x^3-x^2-4x+24$
$-p^3-p^2-4p+24$--- (2)
$-p^3-7p+6=-p^3-p^2-4p+24$
$p^2+3p-18=0$
$(p+6)(p-3)=0$
$p=3$ or $p=-6$

b) When $3x^3+px^2+qx+8$ is divided by x^2-3x+2, the remainder is $5x+6$. Find the values of p and q.

$f(x)=3x^3+px^2+qx+8$
$\quad =(x-2)(x-1)Q(x)+5x+6$
When $x=2$,
$24+4p+2+8=10+6$
$4p+2q=-16$
$2p+q=-8$--- (1)
When $x=1$,
$3+p+q+8=5+6$
$p+q=0$--- (2)
$(1)-(2)$
$p=-8$
Sub p into (2)
$-8+q=0$
$q=8$
$\therefore p=-8, q=8$

c) Find the value for p in which $x^2+5px+p^2+5$ has a factor of $x+2$ but not $x+3$.

$f(x)=x^2+5px+p^2+5$
$x+2$ is a factor
$f(-2)=0$
$4-10p+p^2+5=0$
$p^2-10p+9=0$
$(p-9)(p-1)=0$
$p=9$ or $p=1$
$x+3$ is not a factor
$f(-3)\neq 0$
When $p=9$,
$9-135+81-5=-40$

When $p = 1$,
$9 - 16 + 1 + 5 = 0$
$\therefore p = 9$

Worksheet 3

Summary

- Factorize cubic equations
- Solve cubic equations using Remainder and Factor theorem

Formulae

→All cubic equations have at least 1 root
→Use Trial and Error to find 1 linear factor of the cubic equation

Examples

a) The expression $px^3 - 5x^2 + qx + 10$ has a factor $2x - 1$ but leaves a remainder of -20 when divided by $x + 2$. Find the values of p and q and factorize the expression completely.

$f(x) = px^3 - 5x^2 + qx + 10$
$2x - 1$ is a factor
$f(\frac{1}{2}) = 0$
$\frac{p}{8} - \frac{10}{8} + \frac{4q}{8} + \frac{80}{8} = 0$
$p - 10 + 4q + 80$
$p + 4q = -70$--- (1)
When ÷ by $x + 2$,
$f(-2) = -20$
$-8p - 20 - 2q + 10 = -20$
$8p + 2q = 10$
$4p + q = 5$---(2)
$4 \times (2) - (1)$
$15p = 90$
$p = 6$
Sub p into (2)
$24 + q = 5$
$q = -19$
$6x^3 - 5x^2 - 19x + 10 = (2x - 1)(3x^2 - x - 10)$
$= (3x + 5)(x + 2)$
$\therefore 6x^3 - 5x^2 - 19x + 10 = (2x - 1)(3x + 5)(x + 2)$

b) The expression $x^{2n}+x^3-6x^2-4x+p$ has a factor $(x+2)^2$ and leaves a remainder of 6 when divided by $x+1$. Calculate the value of p and of n. Hence, factorize the equation completely.

$f(x)=x^{2n}+x^3-6x^2-4x+p$
$(x+2)^2$ is a factor
$f(-2)=0$
$4^n-8-24+8+p=0$
$4^n+p=24$ --- (1)
When ÷ by $x+1$,
$f(-1)=6$
$1^n-1-6+4+p=6$
$p=8$
Sub p into (1)
$4^n+8=24$

$4^n=16$
$n=2$
$x^4+x^3-6x^2-4x+8=(x+2)^2(x^2-3x+2)$
$=(x+2)^2(x-2)(x-1)$

Worksheet 4

Summary

- Recall appropriate form for expressing rational functions in partial fractions
- Express rational functions in partial fractions

Formulae

→For each linear factor of the form $(ax+b)$ in the denominator, there will be a partial fraction of the form $\frac{A}{(ax+b)}$ where A is a constant.

→For each linear factor repeated of the form $(ax+b)^2$ in the denominator, there will be partial fractions of the form $\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}$ where A and B are constant.

→For each quadratic factor of the form (x^2+c^2) in the denominator which cannot be factorized, there will be a partial fraction of the form $\frac{A+B}{(x^2+c^2)}$ where A and B are constant.

→For improper fractions, divide the numerator by the denominator to find quotient and new numerator.

→Remember to factorize the denominator completely
→Do not expand the denominator in the answer

Examples

a) Express $\frac{x^3+x-1}{x^2+x^4}$ in partial fractions

$=\frac{x^3+x-1}{x^2(1+x^2)}$

$=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{(1+x^2)}$ (split into partial fractions using the rules)

$x^3+x-1=Ax(1+x^2)+B(1+x^2)+(Cx+D)x^2$

When $x=0$,

$B=-1$

When $x=1$,

$1=2A-2+C+D$

$3=2A+C+D$ --- (1)

When $x=-1$,

$-3=-2A-2-C+D$

$2A+C-D=1$ --- (2)

(1) − (2)

$2D=2$

$D=1$

Compare coefficients of x

$A=1$

Compare coefficients of x^3

$1=1+C$

$C=0$

$\therefore A=1, B=-1, C=0, D=1$

$\frac{x^3+x-1}{x^2(1+x^2)}=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{1+x^2}$

b) $\frac{x^3+2x^2-x+1}{(x-1)(x+2)}$ (improper fraction, so must find quotient and new numerator)

$[x^3+2x^2-x+1]\div[(x-1)(x+2)]=x+1+\frac{3}{(x-1)(x+2)}$

$=\frac{A}{(x-1)}+\frac{B}{(x+2)}$

$x^3+2x^2-x+1=A(x+2)+B(x-1)$

When $x = -2$,

$-8 + 8 + 2 + 1 = -3B$

$3B = -3$

$B = -1$

When $x = 1$,

$1 + 2 - 1 + 1 = 3A$

$3A = 3$

$A = 1$

$$\therefore \frac{x^3 + 2x^2 - x + 1}{(x-1)(x+2)} = x + 1 + \frac{1}{(x-1)} - \frac{1}{(x+2)}$$

Topic Summary

- This topic is basically about how you can manipulate the function using the theorems
- When doing “find A, B and C” questions, always try to substitute x with a suitable value that lets you focus on one of the unknowns
- Keep in mind the theorems and the rules for partial fractions when doing these types of questions

Topic 4: Coordinate Geometry

Worksheet 1

Summary

- Derive and apply formulae for distance and midpoint of 2 given points
- Find the coordinates of the fourth point of a parallelogram

Formulae

→Gradient: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$

→Gradient of parallel lines: If $l_1 \quad l_2$, then $m_1 = m_2$. Parallel lines have equal gradients. The converse is true.

→Gradient of perpendicular lines: If $l_1 \perp l_2$, then $m_1 \perp -\frac{1}{m_2}$, $m_2 \perp -\frac{1}{m_1}$ and $m_1 m_2 = -1$. Perpendicular lines have negative reciprocal gradients. The converse is true.

→Gradient of collinear points: If points A, B and C are collinear, then gradient of AB = the gradient of BC = gradient of AC

→Given any straight line l, let θ be the angle that line l makes with the x-axis. Then gradient $= \tan\theta$.

→Distance between 2 points: $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

→Coordinates of midpoint: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Worksheet 2

Summary

- Solve problems using equation of straight lines
- Derive and apply formulae for area of plane figures

Formulae

→Gradient-intercept form: $y = mx + c$

→Special cases:

a) x-axis: $y = 0$
b) y-axis: $x = 0$
c) Horizontal line: $y = k$ (k is a constant)
d) Vertical line: $x = k$ (k is a constant)
e) Line through origin: $y = mx$

→The coordinates of every point on a straight line will satisfy the equation of the straight line.

→Area of plane figures: For any 3 points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, Area of $ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$

$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$

a) The points are taken in an anti-clockwise direction
b) The formula can be extended to other programs

Worksheet 3

Summary

- Apply the ratio theorem

Formulae

→ Given any 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$, if $P(x, y)$ is a point on AB and it divides AB in the ratio $m : n$, then the coordinates of P is $\left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$

→Quadrilateral: 4 sided polygon
→Parallelogram: Quadrilateral + Diagonals bisect (same midpoint)
→Rhombus: Parallelogram + Diagonals perpendicular
→Rectangle: Parallelogram + One pair of adjacent sides perpendicular
→ Square: Rhombus + One pair of adjacent sides perpendicular OR Rectangle + Diagonals perpendicular

Worksheet 4

Summary

- Write down equation of circle
- Determine the coordinates of the centre and radius of a circle given its equation
- Solve problems involving a circle

Formulae

→Equation of circle: $(x-a)^2 + (r-b)^2 = r^2$
→Coordinates of the centre: (a, b)
→Radius of the circle: r
→General form of equation of circle: $x^2 + y^2 - 2fx - 2gy + c = 0$
Use completing the square to find the coordinates of the centre and the radius of the circle
→Tangent: The tangent to the circle at the point P is a straight line l meeting the circle at P, such that l is perpendicular to the radius of the circle at that point OP

Examples

a) Is the point $P(-1,-2)$ inside, outside or on the circle with equation $(x-1)^2+(y+3)^2=4$

Radius: $\sqrt{4}=2$

Distance from centre to $P=\sqrt{(1+1)^2+(-3+2)^2}$

$=\sqrt{5}$

$\sqrt{5}>2$ (since distance from P is greater than the radius length,)

$\therefore P$ is outside of the circle

b) A circle has equation $x^2+y^2-4x+10y+20=0$. Find the coordinates of the centre and the radius of the circle

$x^2-4x+y^2+10y=-20$

$x^2-4x+4+y^2+10y+25=-20+4+25$

$(x-2)^2+(y+5)^2=9$

Centre is $(2,-5)$

Radius is 3.

c) A circle has equation $x^2+y^2-4x-6=0$. Find the equation of the tangent to the circle at the point $P(1,-3)$.

$x^2-4x+4+y^2=6+4$

$(x-2)^2+y^2=10$

Centre is $(2,0)$

Gradient of OP : $\frac{0-(-3)}{2-1}=3$

Gradient of tangent: $-\frac{1}{3}$

$-3=-\frac{1}{3}+c$

$c=-2\frac{2}{3}$

$y=-\frac{1}{3}x-2\frac{2}{3}$

Topic Summary

- For this topic, all you have to do is to be very familiar with all the formulae

Topic 5: Graphical Solutions of Equations

Worksheet 1

Summary

- Sketch graphs of various equations
- Look at the last page of the worksheet for a summary (too lazy to draw out all the graphs☹)

Worksheet 2

Summary

- Solve equations and inequalities with graphs
- Find gradient at a point
- Define and sketch modulus graphs

Formulae

→To find gradient at a point, draw a line at that point that matches the best with the curve of the graph and find the gradient of that line
→To find values of x, draw the line and look for the intersections
→Read off graph to find x and y -intercepts and also greatest/smallest value of y
→For modulus graphs, sketch the normal graph without the modulus function first, then reflect the part below the x -axis about the x -axis to obtain the modulus graph

Topic 6: Exponential and Logarithmic Equations and Functions

Worksheet 1

Summary

- Solve basic exponential equations

Formulae

→ Laws of indices

1) $a^m \times a^n = a^{m+n}$
2) $a^m \div a^n = a^{m-n}$
3) $(a^m)^n = a^{mn}$
4) $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
5) $a^m \times b^m = (ab)^m$
6) $a^{-m} = \frac{1}{a^m}$
7) $a^0 = 1 (a \neq 0)$

Examples

a) Solve $9^{x+1} - 28(3)^3 + 3 = 0$

$9^{x+1} - 28(3)^3 + 3 = 0$

$3^{3x+2} - 28(3^x) + 3 = 0$ (modify equation so that 3^x is like x in normal quadratic equations)

$9(3^x)^2 - 28(3^x) + 3 = 0$

When $y = 3^x$,

$9y^2 - 28y + 3 = 0$

$(9y - 1)(y - 3) = 0$

$y = \frac{1}{9}$ or $y = 3$

$3^x = \frac{1}{9}$ or $3^x = 3$

$x = -2$ or $x = 1$

b) $2^x + 2^y = 9$ --- (1)

$3^{x-1} = 9 \times 3^y$ --- (2)

From (2),

$3^x \times 3^{-1} = 9 \times 3^y$

$3^{x-1} = 3^{y+2}$

$x - 1 = y + 2$

$x = y + 3$ --- (3)

Sub x into (1)

$2^{y+3} + 2^y = 9$

$8(2^y) + 2^y = 9$

$9(2^y) = 9$

$2^y = 1$

$y = 0$

Sub y into (3),

$x = 3$

$\therefore x = 3, y = 0$

Worksheet 2

Summary

- Derive logarithmic laws
- Convert exponential form to logarithmic form

Formulae

→ $y = a^x$ → $\log_a y = x$ (only when $y > 0, a > 0$ and $a \neq 1$)

→ Laws of logarithms

1) $\log_a xy = \log_a x + \log_a y$
2) $\log_a \frac{x}{y} = \log_a x - \log_a y$
3) $\log_a x^n = n\log_a x$
4) $\log_a a = 1$
5) $\log_a 1 = 0$
6) $a^{\log_a x} = x$

→ Common logarithms: $\log_{10} x = \lg x$

→ Natural logarithms: $\log_e x = \ln x$

Examples

a) $\log_x (4x-3) = 2$

$4x - 3 = x^2$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$

b) $\log_3 12 + \log_3 4 - 2\log_3 \frac{4}{3}$

$= \log_3 48 - 2\log_3 \frac{4}{3}$

$= \log_3 48 - \log_3 \frac{16}{9}$

$= \log_3 \frac{48}{\frac{16}{9}}$ (same base then combine logs)

$= \log_3 27$

$= 3$

c) $3\ln x = \lg 5$

$$\ln x = \frac{\lg 5}{3}$$

$$x = e^{\frac{\lg 5}{3}}$$
$$= 1.26$$

<u>Worksheet 3</u>

Summary

- Solve logarithmic equations

Formulae

→ $\log_a x = \log_a y$ ➔ $x = y$

Examples

a) $(\log_{12} x)^2 = 2\log_{12} x$

$\log_{12} x = y$

$y^2 = 2y$

$y = 0$ or $y = 2$

$\log_{12} x = 0$ or $\log_{12} x = 2$

$x = 12^0$ or $x = 12^2$

$x = 1$ or $x = 144$

b) $\log_9 xy = \frac{3}{2}$ --- (1)

$\log_3 x \log_3 y = -4$ --- (2)

From (1),

$xy = 9^{\frac{3}{2}}$

$xy = 3^3$

$\log_3 xy = 3$

Let $\log_3 x$ be a and let $\log_3 y$ be b

$a + b = 3$

$ab = -4$

$b(3 - b) = -4$

$3b - 3b^2 = -4$

$b^2 - 3b - 4 = 0$

$(b+1)(b-4) = 0$

$b = -1$ or $b = 4$

$a = 4$ or $a = -1$

$\log_3 x = 4$ or $\log_3 x = -1$

$x = 3^4$ or $x = 3^{-1}$

$x = 81$ or $x = \frac{1}{3}$

$\log_3 y = -1$ or $\log_3 y = 4$

$y = 3^{-1}$ or $y = 3^4$

$y = \frac{1}{3}$ or $y = 81$

Worksheet 4

Summary

- Convert base of logarithms

Formulae

→ To change base of logarithms: $\log_a b = \frac{\log_c b}{\log_c a}$ or $\log_a b = \frac{1}{\log_b a}$

Examples

a) Solve $3\log_2 x - 10\log_x 2 + 13 = 0$

$3\log_2 x - \frac{10}{\log_2 x} + 13 = 0$ (Change base)

Let $\log_x 2$ be y

$3y - \frac{10}{y} + 13 = 0$

$3y^2 + 13y - 10 = 0$

$(3y-2)(y+5) = 0$

$y = \frac{2}{3}$ or $y = -5$

$\log_2 x = \frac{2}{3}$ or $\log_2 x = -5$

$x = 2^{\frac{2}{3}}$ or $x = 2^{-5}$

$x = \sqrt[3]{4}$ or $x = \frac{1}{32}$

Worksheet 5

Summary

- Sketch exponential and logarithmic graphs
- Solve exponential and logarithmic equations graphically
- Read through the worksheet to revise (once again, too lazy to draw out graphs)

Topic Summary

- For logarithmic questions, just keep in mind what you can and cannot do with logs
- Always try to manipulate the expression or equation so that the base is the same
- Then, you will be able to combine the log terms using the laws of logarithms
- Finally, remember to reject any answer that will result in a negative term in the log

Topic 7: Quadratic Functions

Worksheet 1

Summary

- Solve algebraic inequalities

Formulae

→Change the inequality sign if multiplying both sides by negative term
→Double check using graph to determine whether answer is in $a < x < b$ or $x < a, x > b$ form
→Modulus:

$|x| = a$

$x = \pm a$

If $|x| \leq a$

$x \leq a$ or $x \geq -a$

If $|x| \geq a$

$x \geq a$ or $x \leq -a$

Worksheet 2

Summary

- Determine nature of roots of quadratic equation based on discriminant
- Use properties of discriminant to solve for unknown constants in an equation

Formulae

→For the quadratic equation $ax^2 + bx + c = 0$, its roots are

$x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

→The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$

→For the equation to have:

i. 2 real and distinct roots: $b^2 - 4ac > 0$

ii. 2 real and equal roots: $b^2 - 4ac = 0$

iii. No real roots: $b^2 - 4ac < 0$

→If a quadratic equation has real roots, then $b^2 - 4ac \geq 0$

→Given any quadratic function $y = ax^2 + bx + c$, when $b^2 - 4ac < 0$

i. If $a > 0$, the curve lies entirely above the x-axis and the function is always positive

ii. If $a < 0$, the curve lies entirely below the x-axis and the function is always negative

Examples

a) Find the range of values of k such that the curves $y = (k-1)x^2 - kx + 1$ and $y = x^2 - 3kx - k$ intersect at exactly 2 points

$x^2 - 3kx - k = kx^2 - x^2 - kx + 1$

$(k-2)x^2 + 2kx + k + 1 = 0$

For 2 points of intersection, discriminant must be > 0

$4k^2 - 4k^2 + 4k + 8 > 0$

$4k + 8 > 0$

$4k > -8$

$k \neq 2$, or the graph will be linear

$\therefore k > -2; k \neq 2$

b) Show that the expression $-2x^2 + 5x - 7$ is negative for all real values of x

For this question, since it stated 'Show', do not start your presentation by stating that discriminant < 0. Instead,

Discriminant: $5^2 - 4(-2)(-7)$

$= 25 - 56$

$= -31$

$-31 < 0$

$\therefore -2x^2 + 5x - 7$ is negative for all real values of x

Worksheet 3

Summary

- Find the maximum or minimum value of a quadratic equation by completing the square

Formulae

→In $y = a(x-h)^2 + k$ form, h is maximum/minimum y value and k is corresponding value of x
→To change a quadratic function into $y = a(x-h)^2 + k$ form:
e.g

$y = x^2 + 2x - 5$

$y = x^2 + 2x + 1 - 5 - 1$

$y = (x+1)^2 - 6$

Minimum point is $(-1, -6)$

Worksheet 4

Summary

- Sketch the quadratic function expressed in the form $y = a(x-h)^2 + k$
- Read through the worksheet to revise

Worksheet 5

Summary

- Apply concept of the sum and product of the roots of a quadratic equation

Formulae

→In the quadratic equation $ax^2 + bx + c = 0$,
the sum of roots $\alpha + \beta = -\frac{b}{a}$,
and the product of roots $\alpha\beta = \frac{c}{a}$
→Useful identities:

$(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$(\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$

$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = (\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right] = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$

$\alpha^3 - \beta^3 = (\alpha-\beta)^3 - 3\alpha\beta(\alpha-\beta) = (\alpha-\beta)\left[(\alpha-\beta)^2 - 3\alpha\beta\right] = (\alpha-\beta)(\alpha^2 + \alpha\beta + \beta^2)$

Examples

a) One root of the equation $2x^2 - x + c = 0$ is twice the other. Find the value of c.

Let roots be α and 2α

Sum of roots: $\alpha + 2\alpha = 3\alpha = \frac{1}{2}$

Product of roots: $2\alpha^2 = \frac{c}{2}$

$2\alpha^2 = \frac{c}{2}$

$3\alpha = \frac{1}{2}$

$\alpha = \frac{1}{6}$

$\frac{c}{2} = 2\left(\frac{1}{6}\right)^2$

$c = \frac{1}{9}$

b) The roots of the equation $3x^2 + 5x + 1 = 0$ are α and β while the roots of the equation $hx^2 - 4x + k = 0$ are $\alpha + 3$ and $\beta + 3$. Calculate the value of h and k.

Sum of roots: $\alpha + \beta = -\frac{5}{3}$

Product of roots: $\alpha\beta = \frac{1}{3}$

Sum of roots: $\alpha + 3 + \beta + 3 = -\frac{5}{3} + 6 = \frac{13}{3}$

Product of roots: $(\alpha + 3)(\beta + 3) = \alpha\beta + 3(\alpha + \beta) + 9 = \frac{1}{3} - 5 + 9 = \frac{13}{3}$

$$\frac{4}{h} = \frac{13}{3}$$
$$12 = 13h$$
$$h = \frac{12}{13}$$
$$\frac{k}{h} = \frac{13}{3}$$
$$3k = 12$$
$$k = 4$$

Topic Summary

- This topic focuses on using discriminant to determine something about a quadratic function
- When doing sum and product of roots questions, manipulate the expression to express it in terms of $\alpha + \beta$ and $\alpha\beta$, then substitute the values in
- Finally, for the graphs, just remember to include all the intercepts and important coordinates (like turning point)

Topic 8: Trigonometry I

Worksheet 1

Summary

- Find the basic angle
- Find trigonometric ratios of angles
- State ratios of special angles $30°, 45°, 60°, 90°, 120°$ either as a fraction or surd form
- Solve trigonometrical equations

Formulae

→ $\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$

→ $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$

→ $\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}}$

→ $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

→In the 1st quadrant, where $0° < \theta < 90°$, **A**ll positive

→In the 2nd quadrant, where $90° < \theta < 180°$, **S**ine positive, Cosine and Tangent negative

→In the 3rd quadrant, where $180° < \theta < 270°$, **T**angent positive, Sine and Cosine negative

→In the 4th quadrant, where $270° < \theta < 360°$, **C**osine positive, Sine and Tangent negative

→ $\sin\theta = \cos(90° - \theta)$

→ $\cos\theta = \sin(90° - \theta)$

→Trigonometric ratios of special angles

θ	30	45	60
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

→When answer is not exact, for angles in degrees, round off to 1 decimal place

Examples

a) Solve $\cos\theta = 0.5$ $(0° < \theta < 360°)$

$\cos\theta = 0.5$ (reduce to basic form, where trigo ratio (angle) = value

θ is in 1st or 4th quadrant (determine quadrant using sign of value)
Basic angle $\alpha = \cos^{-1} 0.5 = 60$ (determine basic angle, ignore sign and multiple angles)
$0 < \theta < 360$ (Check range of angles)
$\theta = 60$ or $\theta = 300$ (Since angle is in 1st or 4th quadrant, there are 2 angles)

Worksheet 2

Summary

- Apply sine rule, cosine rule and find area of triangle using $\frac{1}{2}ab\sin C$

Formulae

→In ABC, the notation of the sides opposite $\angle A$, $\angle B$ and $\angle C$ are a, b and c
→Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
→Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$
$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
→Area of triangle: $\frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$

Worksheet 2

Summary

- Solve practical problems using bearing and trigonometry

Formulae

→Bearings always have 3 digits (i.e. Bearing of 5 is 005)
→Bearings are measured from the north in a clockwise direction

Topic Summary

- In this topic, the most important things are probably the formulae, ASTC and the special angles
- Remember to take note of the range that is specified and change it when required

Topic 9: Trigonometry II

Worksheet 1

Summary

- Convert angles from degrees to radians
- Simplify trigonometrical ratio of negative angles

Formulae

→ $\pi \text{rad} = 180$

→To convert radians to degrees: $\theta \text{rad} = \frac{180\theta}{\pi}$

→To convert degrees to radians: $\theta = \frac{\pi\theta \text{rad}}{180}$

→ When answer is not exact, for angles in radians, round off to 3 significant places

→Negative angles:
$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

→For $\sin^{-1}\theta$ to exist, the domain for $\sin\theta$ is $(-90 \le \theta \le 90)$

→For $\cos^{-1}\theta$ to exist, the domain for $\cos\theta$ is $(0 \le \theta \le 180)$

→For $\tan^{-1}\theta$ to exist, the domain for $\tan\theta$ is $(-90 < \theta < 90)$

→ $\cot\theta = \frac{1}{\tan\theta}$

→ $\sec\theta = \frac{1}{\cos\theta}$

→ $\text{cosec}\,\theta = \frac{1}{\sin\theta}$

→
$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$
$$\frac{\cos\theta}{\sin\theta} = \cot\theta$$

Worksheet 2

Summary

- Solve trigonometrical equations involving multiple angles and common factor
- Solve quadratic trigonometrical equations

Formulae

→Remember to check the range of every trigonometry question
→Remember to use exact form for special angles

Examples

a) Solve $\cos(\theta + 30\ ) = 0.35$

$\cos(\theta + 30\ ) = 0.35$ (reduce to basic form)
θ is in 1st or 4th quadrant
Basic angle $\alpha = \cos^{-1} 0.35 = 69.51$
$30\ \le \theta + 30\ \le 390$
$\theta + 30\ = 69.51$ or $\theta + 30\ = 290.49$
$\theta = 39.5$ or $\theta = 260.5$

b) Solve $\sin 2\theta = -0.8$

$\sin 2\theta = -0.8$ (reduce to basic form)
θ is in 3rd or 4th quadrant
Basic angle $\alpha = \sin^{-1} 0.8 = 53.13$
$0\ \le 2\theta \le 720$
$2\theta = 233.13$ or 306.87 or 593.13 or 666.87
$\theta = 116.6$ or 153.4 or 296.6 or 333.4

Worksheet 3

Summary

- Solve trigonometric equations using Fundamental Identities
- Prove trigonometric identities using Fundamental Identities

Formulae

→ $\sin^2\theta + \cos^2\theta = 1$
→ $1 + \tan^2\theta = \sec^2\theta$
→ $1 + \cot^2\theta = \mathrm{cosec}^2\theta$

Examples

a) $2\cos x - \sin x = \dfrac{3}{2\cos x + \sin x}$ $\qquad (0 \le x \le 2\pi)$

$(2\cos x - \sin x)(2\cos x + \sin x) = 3$
$4\cos^2 x - \sin^2 x = 3$
$4 - 4\sin^2 x - \sin^2 x = 3$

$$5\sin^2 x = 1$$

$$\sin^2 x = \frac{1}{5}$$

$$\sin x = \pm\frac{\sqrt{5}}{5}$$

$$x = 0.464, 3.61, 2.68, 5.82$$

b) $$\frac{1-\sin x}{\cos x} + \frac{\cos x}{1-\sin x} \equiv 2\operatorname{cosec} x \tan x$$

$$\begin{aligned}
&LHS \\
&= \frac{1-\sin x}{\cos x} + \frac{\cos x}{1-\sin x} \\
&= \frac{1-2\sin x + \sin^2 x + \cos^2 x}{\cos x - \sin x \cos x} \\
&= \frac{1-2\sin x + 1}{\cos x - \sin x \cos x} \\
&= \frac{2(1-\sin x)}{\cos x(1-\sin x)} \\
&= \frac{2}{\cos x} \\
&RHS \\
&= 2 \times \frac{1}{\sin x} \times \frac{\sin x}{\cos x} \\
&= \frac{2\sin x}{\sin x \cos x} \\
&= \frac{2}{\cos x} \\
&= LHS
\end{aligned}$$

Worksheet 4

Summary

- Solve trigonometry equations using Addition Formulae
- Prove trigonometry identities using Addition Formulae

Formulae

→ Addition Formulae:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \quad \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \quad \tan A \tan B}$$

→To express $a\cos x - b\sin x$ in the form $R\cos(x+a)$

→ $a = R\cos x$ --- (1)

→ $b = R\sin x$ --- (2)

→ $\tan x = x\dfrac{b}{a}$

→ $R = \pm\sqrt{a^2 + b^2}$

→ $a\cos x - b\sin x$ can be expressed as $R\cos(x+a)$

→ $a\sin x - b\cos x$ can be expressed as $R\sin(x-a)$

→ $a\sin x + b\cos x$ can be expressed as $R\sin(x+a)$ or $R\cos(x-a)$

→IMPORTANT: You have to derive (1) and (2) for every question. Working included in the examples.

Examples

a) $\cos(A+B)\cos(A-B) \equiv \cos^2 A - \sin^2 B$

$$
\begin{aligned}
& LHS \\
&= \cos(A+B)\cos(A-B) \\
&= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B) \\
&= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B \\
&= \cos^2 A(1-\sin^2 B) - \sin^2 B(1-\cos^2 A) \\
&= \cos^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B \\
&= \cos^2 A - \sin^2 B \\
&= RHS
\end{aligned}
$$

b) Solve $3\sin\theta - \cos\theta = 2$

Let $3\sin\theta - \cos\theta = R\sin(\theta-\alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$R\cos\alpha = 3$ --- (1)

$R\sin\alpha = 1$ --- (2)

$(1)^2 + (2)^2 : R^2 = 3^2 + 1^2$

$R = \sqrt{10}$

$(2) \div (1) : \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{1}{3}$

$\tan\alpha = \dfrac{1}{3}$

$\alpha = 18.43$

$$3\sin\theta - \cos\theta = \sqrt{10}\sin(\theta - 18.43\ ) = 2$$

$$\sin(\theta - 18.43\ ) = \frac{2}{\sqrt{10}}$$

$$\theta - 18.43\ = 39.23\ , 140.77$$

$$\theta = 57.7\ , 159.2$$

Worksheet 5

Summary

- Find trigonometrical ratios using Multiple Angle Formulae
- Solve equations using Multiple Angle Formulae
- Prove identities using Multiple Angle Formulae

Formulae

Double Angle Formulae:

→ $\sin 2A = 2\sin A\cos A$

→ $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

→ $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Half Angle Formulae

→ $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

→ $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = 1 - 2\sin^2\dfrac{A}{2} = 2\cos^2\dfrac{A}{2} - 1$

→ $\tan A = \dfrac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$

Examples

a) Express $\cos 3x$ in terms of $\cos x$. Hence solve the equation $\cos 3x + \cos^2 x = 0$ for $0\ < x < 360$

$$\begin{aligned}
&\cos 3x \\
&= \cos(x + 2x) \\
&= \cos x\cos 2x - \sin x\sin 2x \\
&= \cos(2\cos^2 x - 1) - \sin x(2\sin x\cos x) \\
&= 2\cos^3 x - \cos x - 2\sin^2 x\cos x \\
&= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x \\
&= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x
\end{aligned}$$

$= 4\cos^3 x - 3\cos x$

$4\cos^3 + \cos^2 x - 3\cos x = 0$

$\cos x(4\cos^2 + \cos x - 3) = 0$

$\cos x(4\cos x - 3)(\cos x + 1) = 0$

$\cos x = \frac{3}{4}$ or $\cos x = -1$ or $\cos x = 0$

$x = 41.4\ ,318.6\ ,180\ ,90\ ,270$

b) Given that $\cos B = -\frac{12}{13}$ and that B is in the 3rd quadrant, find the value of $\cos\frac{B}{2}$ without the use of calculators

$\cos B = 2\cos^2\frac{B}{2} - 1$

$2\cos^2\frac{B}{2} = \cos B + 1$

$\cos^2\frac{B}{2} = \frac{\cos B + 1}{2}$

$\cos\frac{B}{2} = \pm\sqrt{\frac{\cos B + 1}{2}}$

Since $180\ < B < 270$

$90\ < \frac{B}{2} < 135$

$\frac{B}{2}$ is in the 2nd quadrant

$\therefore \cos\frac{B}{2} = -\sqrt{\frac{\cos B + 1}{2}}$

$= -\sqrt{\frac{\left(-\frac{12}{13}\right) + 1}{2}}$

$= -\sqrt{\frac{1}{26}}$

$= -\frac{1}{\sqrt{26}}$

$= -\frac{\sqrt{26}}{26}$

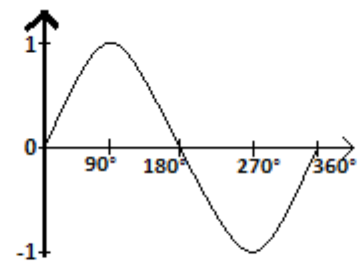
Worksheet 6

Summary

- Sketch trigonometric graphs, noting the period and amplitude, in the form $y = a\sin(bx) + c$ and cosine and tangent
- Solve equations using trigonometric graphs

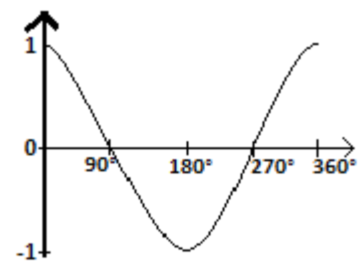
Formulae

y = sinx Graph



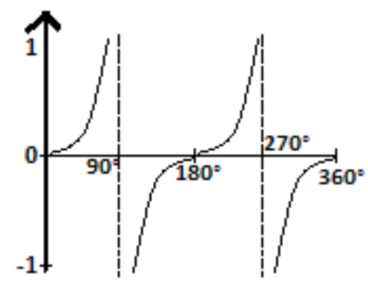
→Since range of $\sin x$ is $-1 \le \sin x \le 1$, amplitude of $y = \sin x$ is 1
→ x -intercepts are $0\ , 180\ , 360$ or $0, \pi, 2\pi$ and period is 360 or 2π

y = cosx Graph



→Since range of $\cos x$ is $-1 \le \cos x \le 1$, amplitude of $y = \cos x$ is 1
→ x -intercepts are $90\ , 270$ or $\frac{\pi}{2}, \frac{3\pi}{2}$

y=tanx Graph



→Since range of $\tan x$ is all real numbers, no min/max value
→ x -intercepts are $0\ , 180\ , 360$ or $0, \pi, 2\pi$ and period is 180 or π
→Vertical asymptotes exist at $90\ , 270$ or $\frac{\pi}{2}, \frac{3\pi}{2}$

→In $y = a\sin(bx) + c$:
→ The a value modifies the amplitude of the graph
- The amplitude of $y = a\sin x$ and $y = a\cos x$ is a , and thus the range is $-a \le a\sin x \le a$
- As such, the markings on the side is changed from -1 and 1 to –a and a
- Note that $y = \tan x$ has no range and amplitude, thus a does not affect $\tan x$ graphs

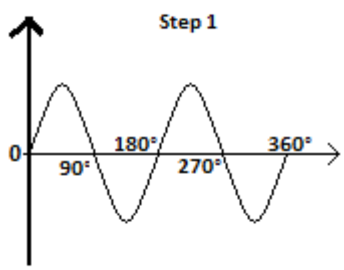
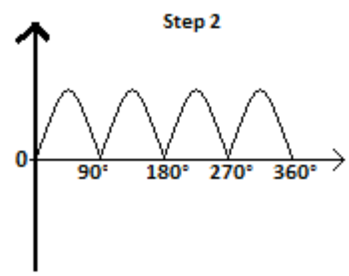
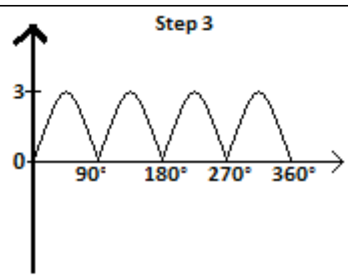
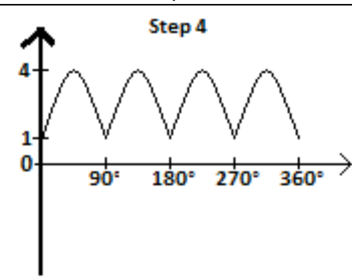
→ The b value modifies the period of the graph
- The period of $y = \sin bx$, $y = \cos bx$ and $y = \tan bx$ is $\frac{\text{Original period}}{b}$
- As such, the number of waves and also the x-intercepts are changed accordingly

→ The c value modifies where the whole graph is placed
- Move the whole graph up or down accordingly to the c value
- As such, the markings on the side as well as the axis are changed accordingly

→ Note that when sketching $y = a\sin(bx) + c$ graphs, always take b into account first, then a and then finally c in order to save redrawing time. This is because changing the numbers at the side is easier than changing and redrawing the whole graph.

Examples

a) Sketch $y = 3|\sin 2x| + 1$ $(0 \le x \le 360)$

 Step 1: Since $b = 2$, period is $\frac{360}{2} = 180$	 Step 2: $\|\sin 2x\|$, so reflect everything under the x-axis upon the x-axis
 Step 3: Since $a = 3$, amplitude is 3. Note that there is no -3 marking because of modulus	 Step 4: Since $c = 1$, shift everything up by 1. Note the change of the markings

Topic Summary

- When doing trigonometry questions, keep all the formulae in mind and try to see the expression or equation in ways such that it allows you to make use of the formulae
- Always keep in mind the ASTC rule
- Remember that 1 d.p. is not required for special angles
- When the answers are in radians and are special angles, the exact value must be given (usually in terms of π)

<u>Topic 10: Circular Measure</u>

<u>Worksheet 1</u>

Summary

- Calculate length of arc, area of sector and segment
- Solve problems involving circles

Formulae

In all the formulae below, θ is in radians.
→Length of arc: $r\theta$
→Area of sector: $\frac{1}{2}r^2\theta$
→Area of segment: $\frac{1}{2}r^2(\theta - \sin\theta)$

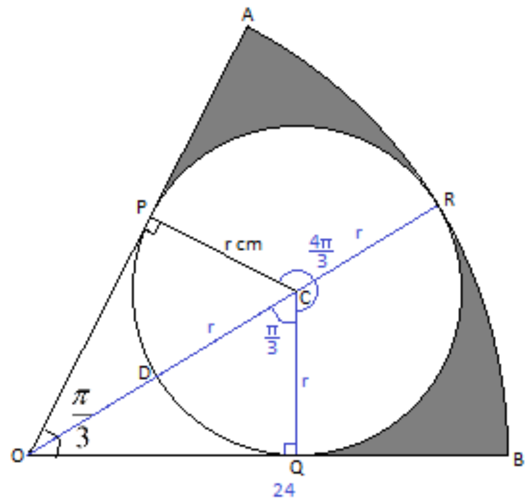
Examples

a) In the figure, OAB is a sector of a circle with centre O and radius 24cm. The circle PQR with centre C and radius r cm is inscribed in the sector.

i) Show that r = 8 and find OQ

ii) Find the area of the shaded region, leaving the answer in the form $a\pi + b\sqrt{3}$

iii) Find the area of the minor segment ARB



$OD + 2r = 24$

$\cos\frac{\pi}{3} = \frac{r}{r + OD}$

$\cos\frac{\pi}{3} = \frac{r}{r + 24 - 2r}$

$\cos\frac{\pi}{3} = \frac{r}{24 - r}$

$\frac{24 - r}{r} = \frac{1}{\cos\frac{\pi}{3}}$

$\frac{24}{r} - 1 = \frac{1}{\cos\frac{\pi}{3}}$

$\frac{24}{r} = \frac{1}{\cos\frac{\pi}{3}} + 1$

$r = \frac{24}{\frac{1}{\cos\frac{\pi}{3}} + 1} = 8$

$OD = 24 - 8 - 8 = 8$

$OQ = \sqrt{8^2 + 16^2}$

$OQ = 8\sqrt{3}$

$OAB = \frac{1}{2}(24)^2\frac{\pi}{3}$

$OAB = 96\pi$

$OQCP = 2 \times \frac{1}{2}(8)(8\sqrt{3})$

$OQCP = 64\sqrt{3}$

$PRQC = \frac{1}{2}(8)^2(\frac{4\pi}{3})$

$PRQC = \frac{128\pi}{3}$

$96\pi - \frac{128\pi}{3} - 64\sqrt{3} = \frac{160\pi}{3} - 64\sqrt{3}$

Segment $ARB = \frac{1}{2}(24)^2\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$

$= 52.2$

Summary Page

Laws of Surds:

→ $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

→ $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

→ $\sqrt{a} \times \sqrt{a} = a$

→ $a\sqrt{c} \times b\sqrt{d} = ab\sqrt{cd}$

→ $a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$

Rationalizing Surds:

→ $\dfrac{a}{\sqrt{b}} = \dfrac{a}{\sqrt{b}} \times \dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a\sqrt{b}}{b}$

→ $\dfrac{a}{b+\sqrt{c}} = \dfrac{a}{b+\sqrt{c}} \times \dfrac{b-\sqrt{c}}{b-\sqrt{c}} = \dfrac{ab-a\sqrt{c}}{(b+\sqrt{c})(b-\sqrt{c})} = \dfrac{ab-a\sqrt{c}}{b^2-c}$

Polynomial form:

→ $P(x) = D(x) \times Q(x) + R(x)$

Remainder Theorem:

→If $f(x)$ is divided by $(ax-b)$, the remainder is $f(\dfrac{b}{a})$

Factor Theorem:

→If $(ax-b)$ is a factor of $f(x)$, then $f(\dfrac{b}{a}) = 0$

Partial Fractions:

→For each linear factor of the form $(ax+b)$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{(ax+b)}$ where A is a constant.

→For each linear factor repeated of the form $(ax+b)^2$ in the denominator, there will be partial fractions of the form $\dfrac{A}{(ax+b)} + \dfrac{B}{(ax+b)^2}$ where A and B are constant.

→For each quadratic factor of the form (x^2+c^2) in the denominator which cannot be factorized, there will be a partial fraction of the form $\dfrac{A+B}{(x^2+c^2)}$ where A and B are constant.

→For improper fractions, divide the numerator by the denominator to find quotient and new numerator.

Gradients:

→Gradient: $\dfrac{y_2-y_1}{x_2-x_1} = \dfrac{y_1-y_2}{x_1-x_2}$

→Gradient of parallel lines: If $l_1 \quad l_2$, then $m_1 = m_2$.

→Gradient of perpendicular lines: If $l_1 \perp l_2$, then $m_1 \perp -\frac{1}{m_2}$, $m_2 \perp -\frac{1}{m_1}$ and $m_1 m_2 = -1$.

→Gradient of collinear points: If points A, B and C are collinear, then gradient of $AB =$ the gradient of $BC =$ gradient of AC

→Given any straight line l, let θ be the angle that line l makes with the x-axis. Then gradient $= \tan\theta$.

Distance between 2 points:

→ $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

Coordinates of midpoint:

→ $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Gradient-intercept form:

→ $y = mx + c$

Area of polygons:

→Area of $ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$

$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$

a) The points are taken in an anti-clockwise direction
b) The formula can be extended to other polygons

Ratio Theorem:

→ Given any 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$, if $P(x, y)$ is a point on AB and it divides AB in the ratio $m:n$, then the coordinates of P is $\left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$

Circles:

→Equation of circle: $(x-a)^2 + (r-b)^2 = r^2$

→Coordinates of the centre: (a, b)

→Radius of the circle: r

→General form of equation of circle: $x^2 + y^2 - 2fx - 2gy + c = 0$

Use completing the square to find the coordinates of the centre and the radius of the circle

→Tangent: The tangent to the circle at the point P is a straight line l meeting the circle at P, such that l is perpendicular to the radius of the circle at that point OP

Graphs:

→To find gradient at a point, draw a line at that point that matches the best with the curve of the graph and find the gradient of that line

→For modulus graphs, sketch the normal graph without the modulus function first, then reflect the part below the x-axis about the x-axis to obtain the modulus graph

Laws of Indices:

1) $a^m \times a^n = a^{m+n}$
2) $a^m \div a^n = a^{m-n}$
3) $(a^m)^n = a^{mn}$
4) $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

5) $a^m \times b^m = (ab)^m$

6) $a^{-m} = \frac{1}{a^m}$

7) $a^0 = 1 (a \neq 0)$

Logarithms:

→ $y = a^x$ ➔ $\log_a y = x$ (only when $y > 0, a > 0$ and $a \neq 1$)

→Laws of logarithms

1) $\log_a xy = \log_a x + \log_a y$
2) $\log_a \frac{x}{y} = \log_a x - \log_a y$
3) $\log_a x^n = n \log_a x$
4) $\log_a a = 1$
5) $\log_a 1 = 0$
6) $a^{\log_a x} = x$

→Common logarithms: $\log_{10} x = \lg x$

→Natural logarithms: $\log_e x = \ln x$

→ $\log_a x = \log_a y$ ➔ $x = y$

→ To change base of logarithms: $\log_a b = \frac{\log_c b}{\log_c a}$ or $\log_a b = \frac{1}{\log_b a}$

Quadratic Functions:

→Change the inequality sign if multiplying both sides by negative term

→Double check using graph to determine whether answer is in $a < x < b$ or $x < a, x > b$ form

→Modulus:

$|x| = a$

$x = \pm a$

If $|x| \leq a$

$x \leq a$ or $x \geq -a$

If $|x| \geq a$

$x \geq a$ or $x \leq -a$

Discriminant:

→For the quadratic equation $ax^2 + bx + c = 0$, its roots are

$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

→The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$

→For the equation to have:

iv. 2 real and distinct roots: $b^2 - 4ac > 0$

v. 2 real and equal roots: $b^2 - 4ac = 0$

vi. No real roots: $b^2 - 4ac < 0$

→If a quadratic equation has real roots, then $b^2 - 4ac \geq 0$

→Given any quadratic function $y = ax^2 + bx + c$, when $b^2 - 4ac < 0$

iii. If $a > 0$, the curve lies entirely above the x-axis and the function is always positive

iv. If $a < 0$, the curve lies entirely below the x-axis and the function is always negative

$y = a(x-h)^2 + k$ form:

→In $y = a(x-h)^2 + k$ form, h is maximum/minimum y value and k is corresponding value of x

→To change a quadratic function into $y = a(x-h)^2 + k$ form:

e.g

$y = x^2 + 2x - 5$

$y = x^2 + 2x + 1 - 5 - 1$

$y = (x+1)^2 - 6$

Minimum point is $(-1, -6)$

Sum and Product of Roots:

→In the quadratic equation $ax^2 + bx + c = 0$,

the sum of roots $\alpha + \beta = -\frac{b}{a}$,

and the product of roots $\alpha\beta = \frac{c}{a}$

→Useful identities:

$(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$(\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$

$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = (\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right] = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$

$\alpha^3 - \beta^3 = (\alpha-\beta)^3 - 3\alpha\beta(\alpha-\beta) = (\alpha-\beta)\left[(\alpha-\beta)^2 - 3\alpha\beta\right] = (\alpha-\beta)(\alpha^2 + \alpha\beta + \beta^2)$

Trigonometry:

→ $\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$

→ $\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$

→ $\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$

→ $\frac{\sin\theta}{\cos\theta} = \tan\theta$

→In the 1st quadrant, where $0 < \theta < 90$, **A**ll positive

→In the 2nd quadrant, where $90 < \theta < 180$, **S**ine positive, Cosine and Tangent negative

→In the 3rd quadrant, where $180 < \theta < 270$, **T**angent positive, Sine and Cosine negative

→In the 4th quadrant, where $270 < \theta < 360$, **C**osine positive, Sine and Tangent negative

→ $\sin\theta = \cos(90 - \theta)$

→ $\cos\theta = \sin(90 - \theta)$

→Trigonometric ratios of special angles

θ	30	45	60
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

→In ABC, the notation of the sides opposite $\angle A$, $\angle B$ and $\angle C$ are a, b and c

Sine rule:

→ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine rule:

$a^2 = b^2 + c^2 - 2bc\cos A$

→ $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Area of triangle:

→ $\frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$

Radians:

→ $\pi\text{rad} = 180$

→To convert radians to degrees: $\theta\text{rad} = \frac{180\theta}{\pi}$

→To convert degrees to radians: $\theta = \frac{\pi\theta\text{rad}}{180}$

Negative angles:

$\sin(-\theta) = -\sin\theta$

→ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

Arc trigonometric ratios:

→For $\sin^{-1}\theta$ to exist, the domain for $\sin\theta$ is $(-90 \le \theta \le 90)$

→For $\cos^{-1}\theta$ to exist, the domain for $\cos\theta$ is $(0 \le \theta \le 180)$

→For $\tan^{-1}\theta$ to exist, the domain for $\tan\theta$ is $(-90 < \theta < 90)$

Special trigonometric ratios:

→ $\cot\theta = \frac{1}{\tan\theta}$

→ $\sec\theta = \frac{1}{\cos\theta}$

→ $\text{cosec}\,\theta = \frac{1}{\sin\theta}$

→ $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

$\dfrac{\cos\theta}{\sin\theta} = \cot\theta$

Fundamental Identities:

→ $\sin^2\theta + \cos^2\theta = 1$

→ $1 + \tan^2\theta = \sec^2\theta$

→ $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Addition Formulae:

→ $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

→ $\cos(A \pm B) = \cos A\cos B \quad \sin A\sin B$

→ $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \quad \tan A\tan B}$

R-Formulae:

→To express $a\cos x - b\sin x$ in the form $R\cos(x + a)$

→ $a = R\cos x$ --- (1)

→ $b = R\sin x$ --- (2)

→ $\tan x = x\dfrac{b}{a}$

→ $R = \pm\sqrt{a^2 + b^2}$

→ $a\cos x - b\sin x$ can be expressed as $R\cos(x + a)$

→ $a\sin x - b\cos x$ can be expressed as $R\sin(x - a)$

→ $a\sin x + b\cos x$ can be expressed as $R\sin(x + a)$ or $R\cos(x - a)$

→IMPORTANT: You have to derive (1) and (2) for every question. Working included in the examples.

→ $\sin 2A = 2\sin A\cos A$

→ $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

→ $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

→ $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

→ $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = 1 - 2\sin^2\dfrac{A}{2} = 2\cos^2\dfrac{A}{2} - 1$

→ $\tan A = \dfrac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$

Trigonometric graphs:

→Since range of $\sin x$ is $-1 \le \sin x \le 1$, amplitude of $y = \sin x$ is 1

→ x-intercepts are 0 ,180 ,360 or $0, \pi, 2\pi$ and period is 360 or 2π

→Since range of $\cos x$ is $-1 \le \cos x \le 1$, amplitude of $y = \cos x$ is 1

→ x -intercepts are $90\ ,270$ or $\frac{\pi}{2},\frac{3\pi}{2}$

→Since range of $\tan x$ is all real numbers, no min/max value

→ x -intercepts are $0\ ,180\ ,360$ or $0,\pi,2\pi$ and period is 180 or π

→Vertical asymptotes exist at $90\ ,270$ or $\frac{\pi}{2},\frac{3\pi}{2}$

→ The a value modifies the amplitude of the graph

→ The amplitude of $y = a\sin x$ and $y = a\cos x$ is a , and thus the range is $-a \le a\sin x \le a$

→ As such, the markings on the side is changed from -1 and 1 to –a and a

→ The b value modifies the period of the graph

→ The period of $y = \sin bx$, $y = \cos bx$ and $y = \tan bx$ is $\frac{\text{Original period}}{b}$

→ As such, the number of waves and also the x-intercepts are changed accordingly

→ The c value modifies where the whole graph is placed

→ Move the whole graph up or down accordingly to the c value

→ As such, the markings on the side as well as the axis are changed accordingly

In all the formulae below, θ is in radians.

→Length of arc: $r\theta$

→Area of sector: $\frac{1}{2}r^2\theta$

→Area of segment: $\frac{1}{2}r^2(\theta - \sin\theta)$

Important Pointers

- All cubic equations have at least 1 root
- When answer is not exact, for angles in degrees, round off to 1 decimal place
- When answer is not exact, for angles in radians, round off to 3 significant places
- Bearings always have 3 digits (i.e. Bearing of 5 is 005)
- Bearings are measured from the north in a clockwise direction
- Remember to reject answers when:
  - Term in square root is negative
  - Term in log is negative
  - Term involves dividing by zero
- For proving questions, make sure you do not skip any steps, as you will be marked down
- Note that when sketching $y = a\sin(bx) + c$ graphs, always take b into account first, then a and then finally c in order to save redrawing time. This is because changing the numbers at the side is easier than changing and redrawing the whole graph.
- When working with circles, radians and degrees, for intermediate steps, try to calculate radians into at least 5 d.p. because a small change in radians will result in a large change in degrees, thus accuracy is required.
- Check your answers after you have finished the paper by substituting the values into the equation